ON THE MATHEMATICAL CONTENT OF VOLTERRA'S PRINCIPLE IN THE BOUNDARY VALUE PROBLEM OF VISCOELASTICITY

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V. G. GROMOV
(Rostov-on-Don)
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A direct proof of Volterra's principle is given by reducing the boundary value problem of homogeneous viscoelasticity to solving the corresponding elastic problem and some operator equations. The conditions of applicability of the symbolic method are formulated as equivalence conditions between the realization of the viscoelastic operator function which arises from the elastic problem and the solutions of the operator equations. We note that the second procedure is more general and can be used in the problems of viscoelasticity whose solutions cannot be constructed by Volterra's principle.

Volterra's principle [1 - 5] is widely used for the construction of the solution of boundary value problems in linear homogeneous viscoelasticity. The basis for its applicability is the independence of the operations with respect to the coordinates and time in the complete fundamental system of quasi-static equations of a viscoelastic body. As a result, the problem is divided into solving the corresponding boundary value problem of the elastic body and the determination of the operator functions. The latter are obtained from the elastic solution through the formal replacement of the mechanical moduli by the viscoelastic operators.

However, the separation of the space and time operations in the equations of viscoelasticity is, by itself, not a sufficient criterion for the applicability of the operator-symbolic method, if only because the boundary conditions are not taken into account.

In connection with this, an investigation of the problem of the rational application of Volterra's principle is required. An attempt for the mathematical foundation of Volterra's principle is contained in [6]. In the case of time-independent viscoelastic properties the identity between the first and second form of the correspondence principle has been established by the methods of operational calculus [7].

The symbolic method is justified by the construction of an isomorphism between the sets of functions of the viscoelasticity operators and the functions of a complex variable. The conditions of applicability of Volterra's principle are determined by the possibility of performing a Laplace transform in the equations and the boundary conditions of the viscoelasticity problem.

1. Two schemes for the construction of the solutions of the boundary value problem. We assume that a viscoelastic body occupies the domain Ω, bounded by the surface S of the Euclidean space, x is a point of the space,

x_i its coordinates, and t denotes time. We denote by $\mathbf{u}(x, t)$ the displacement vector, by u_i its componenets, and by $\varepsilon_{ij}(x, t)$, $\sigma_{ij}(x, t)$ the components of the strain and stress tensors, respectively, in the viscoelastic body. The complete fundamental system of equations of the quasistatic problem of a viscoelastic body has the form

$$\sigma_{ij,j} + f_i = 0, \qquad \sigma_{ij} = \vartheta^\circ_{ijkl}\varepsilon_{kl}, \qquad \varepsilon_{ij} = {}^1/_2(u_{i,j} + u_{j,i}) \tag{1.1}$$

$$\vartheta^\circ_{ijkl} = E_{ijkl} - \vartheta^*_{ijkl} = E_{ijkl} - \int\limits_0^t \vartheta_{ijkl}(t, \tau)(\cdots)\,d\tau \tag{1.2}$$

Here $f_i(x, t)$ are the given body forces per unit of volume, $\overset{\circ}{\vartheta}_{ijkl}$ is the tensor-operator of anisotropic viscoelasticity, E_{ijkl} are elastic constants, and $\vartheta_{ijkl}(t, \tau)$ is the heredity kernel. We remind that the indices after the comma denote differentiation with respect to the corresponding coordinates and repeated indices denote a summation from one to three. For the sake of simplicity we consider the boundary value problem of viscoelasticity in displacements. By eliminating the stresses from (1.1) we obtain the equations for the displacements

$$\vartheta^\circ_{ijkl}u_{k,lj} + f_i = 0 \qquad (x \in \Omega, 0 \leqslant t < \infty) \tag{1.3}$$

Here we have made use of the symmetry of the tensor-operator

$$\vartheta^\circ_{ijkl} = \vartheta^\circ_{jikl} = \vartheta^\circ_{ijlk} \tag{1.4}$$

We formulate the boundary conditions in the form

$$u_i(x, t) = 0 \qquad (x \in S, \ 0 \leqslant t < \infty) \tag{1.5}$$

Arbitrary, but sufficiently smooth boundary conditions are reduced to the form (1.5) by introducing a twice continuously differentiable auxiliary function in the domain Ω. Thus, the problem consists in finding the displacement vector $\mathbf{u}(x, t)$ which satisfies Eqs. (1.3) for each $t \in [0, \infty)$ inside the domain $(x \in \Omega)$, and the conditions (1.5) on the boundary $(x \in S)$.

The first scheme for the construction of the solution of the problem (1.3), (1.5), connected with Volterra's principle, consists in solving first the elastic boundary value problem

$$E_{ijkl}u_{k,lj} + f_i = 0 \quad (x \in \Omega), \quad u_i = 0 \quad (x \in S, \ 0 \leqslant t < \infty) \tag{1.6}$$

The time plays the part of a parameter. Assume that the solution of problem (1.6) has been found, i.e. an operator E^{-1} has been found such that

$$\mathbf{u} = E^{-1}\mathbf{f} \qquad (x \in \Omega + S, \ 0 \leqslant t < \infty) \tag{1.7}$$

and the relations (1.6) are satisfied. Obviously, this operator depends on the numerical parameters which occur in the problem (1.6). The tensor of the elastic constants E_{ijkl} belongs to these in the first place, and possibly the time t,

$$E^{-1} = F(E_{1111}, E_{1122} \ldots E_{3333}, t) \tag{1.8}$$

The equality (1.8) defines an abstract function of several numerical arguments [8]. The derivation of the solution of the viscoelastic problem (1.3), (1.5) consists in the realization (determination) of this function when the elastic constants E_{ijkl} are replaced by the corresponding viscoelastic operators ϑ°_{ijkl}. The realization procedure consists in the formal expansion of the function (1.8) in an abstract Taylor series in the neighborhood of the numerical part of the operators ϑ°_{ijkl}

$$F(\vartheta^\circ_{1111}, \vartheta^\circ_{1122} \cdots \vartheta^\circ_{3333}, t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (\vartheta^*_{ijkl}\partial_{ijkl})^n E^{-1} \qquad (1.9)$$

where the symbol ∂_{ijkl} denotes differentiation with respect to the parameter E_{ijkl}, and

$$\partial_{ijkl}E^{-1} = \lim_{\Delta E_{ijkl}\to 0} \frac{F(E_{1111}, \cdots E_{ijkl} + \Delta E_{ijkl}, \cdots E_{3333}) - F(E_{1111}, \cdots E_{ijkl}, \cdots E_{3333})}{\Delta E_{ijkl}}$$

and the limit is understood in the sense of the norm of the multiparametric space of operators generated by the operator E^{-1}. By the power of the operator $\vartheta^*_{ijkl}\partial_{ijkl}$ one has to understand its repeated application. By virtue of the noncommutativity of the operators ϑ^*_{ijkl} among themselves, one cannot give for the n th degree of this operator a contracted representation similar to the one which holds for numerical functions. Finally, the mixed derivative is a sequence of mixed derivatives of order k of an abstract function with respect to the corresponding aggregate of numerical parameters [8].

The second scheme for solving the boundary problem of viscoelasticity differs from the previous one in that the realization procedure is replaced by solving of the operator equations. We introduce two operators E and ϑ_t by the expressions

$$(E\mathbf{u})_i = -E_{ijkl}u_{k,lj}, \qquad (\vartheta_t\mathbf{u})_i = -\vartheta^*_{ijkl}u_{k,lj} \qquad (1.11)$$

After that the problem (1.3), (1.5) can be represented in vector form

$$E\mathbf{u} - \vartheta_t\mathbf{u} = \mathbf{f} \quad (x \in \Omega), \qquad \mathbf{u} = 0 \quad (x \in S,\ 0 \leqslant t < \infty) \qquad (1.12)$$

For the formation of the operator equation we introduce an auxiliary vector $\mathbf{g}\ (x, t)$ and we consider the elastic problem

$$E\mathbf{u} = \mathbf{g} \quad (x \in \Omega), \quad \mathbf{u} = 0 \quad (x \in S) \qquad (1.13)$$

After solving this problem we obtain the displacement vector

$$\mathbf{u} = E^{-1}\mathbf{g} \qquad (x \in \Omega + S) \qquad (1.14)$$

which satisfies the boundary condition of the initial problem (1.12). Now we select the vector g such that Eqs. (1.12) should hold inside the domain. For this, it should satisfy the equation

$$\mathbf{g} - A_t\mathbf{g} = \mathbf{f} \qquad (x \in \Omega,\ 0 \leqslant t < \infty) \qquad (1.15)$$

where $A_t = \vartheta_t E^{-1}$ is the product of the operators. The formal solution of Eq. (1.15) is represented by the Neumann iterative series [8]

$$\mathbf{g} = (I - A_t)^{-1}\mathbf{f} = \sum_{n=0}^{\infty} A_t^{\,n}\mathbf{f} \qquad (1.16)$$

The solution of the boundary value problem (1.12) takes the form

$$\mathbf{u} = \sum_{n=0}^{\infty} E^{-1}A_t^{\,n}\mathbf{f} = V_t^{-1}\mathbf{f} \qquad (1.17)$$

Subsequently we shall elucidate the conditions under which the expansion (1.9), obtained by Volterra's scheme, is equivalent to the operator V_t^{-1}, and we shall give the proof of this expansion by means of the representation (1.17).

2. The equivalence of the two schemes. We continue for the present the formal examination of the problem. It is convenient to introduce the operator

$$B_t = \sum_{n=0}^{\infty} A_t^n \tag{2.1}$$

Now the inverse operator of the viscoelastic problem is represented by the product

$$V_t^{-1} = E^{-1} B_t \tag{2.2}$$

The proof of the equivalence of the expansion (1.9) with the operator V_t^{-1} reduces to the verification of the commutativity of the operators E^{-1} and B_t

$$E^{-1} B_t = B_t E^{-1} \tag{2.3}$$

and the validity of the identity

$$B_t = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} E \left(\partial^*{}_{ijkl} \partial_{ijkl}\right)^n E^{-1} \tag{2.4}$$

where, as before, the operator E is defined by the boundary value problem (1.13).

Lemma 2.1. For the operator A_t we have the representation

$$A_t = \partial^*{}_{ijkl} \partial_{ijkl} E E^{-1} \tag{2.5}$$

where the operator $\partial_{ijkl} E$ denotes the derivative of the operator E with respect to the parameter E_{ijkl}.

Proof. Let $\mathbf{e}_i$ be the coordinate vectors. Taking into account (1.11), we obtain explicit representations for the operator of the elastic problem

$$E = -\mathbf{e}_i E_{ijkl} (\cdots)_{k,\, lj} \tag{2.6}$$

and for the operator under consideration

$$A_t = -\mathbf{e}_i\, \partial^*_{ijkl} \left(E^{-1} (\cdots)\right)_{k,\, lj} \tag{2.7}$$

Differentiating (2.6) with respect to the parameter E_{ijkl}, we have

$$\partial_{ijkl} E = -\partial_{ijkl} \left[\mathbf{e}_m E_{mnpq} (\cdots)_{p,\, qn}\right] = -\mathbf{e}_i (\cdots)_{k,\, lj} \tag{2.8}$$

since the operator E is a linear function of the parameters E_{ijkl} and, consequently, by differentiation, only those terms will be distinct from zeros whose indices coincide with the differentiation indices. Applying this to (2.7), we obtain the required representation (2.5).

Lemma 2.2. On the set of vectors on which the operators E and E^{-1} are commutative, the operators $\partial_{ijkl} E$ and E^{-1} are commutative.

Proof. The operator E can be represented as a linear combination of operators $\partial_{ijkl} E$

$$E = E_{ijkl}\, \partial_{ijkl} E \tag{2.9}$$

with coefficients E_{ijkl}, among which, by virtue of the symmetry property (1.4), only 36 are independent. Expressing (2.9) in terms of the independent coefficients ε_{ijkl}, we obtain

$$E = \varepsilon_{ijkl}\, \partial_{ijkl} E \tag{2.10}$$

where only 36 out of 81 will be distinct from zero. These coefficients can be written out, but here this is not essential. Under the conditions of commutativity and linearity of the operators E and E^{-1} we have

$$EE^{-1} - E^{-1}E = \varepsilon_{ijkl}(\partial_{ijkl} EE^{-1} - E^{-1}\partial_{ijkl} E) = 0 \tag{2.11}$$

Since ε_{ijkl} are independent, equality (2.11) means that for every vector from the mentioned set, we have

$$(\partial_{ijkl} EE^{-1} - E^{-1}\partial_{ijkl} E)(\cdots) = 0 \tag{2.12}$$

i.e. the operators E^{-1} and $\partial_{ijkl}\ E$ are commutative.

Theorem 2.1. If the conditions of Lemma 2.2 hold and if the operators ϑ^*_{ijkl}, E and E^{-1} are commutative, the identity (2.4) holds.

Proof. For a proof of identity (2.4) it is necessary to show that

$$E(\vartheta^*_{ijkl}\ \partial_{ijkl})^n E^{-1} = (-1)^n n!\,(\vartheta^*_{ijkl}\,\partial_{ijkl}\ EE^{-1})^n \tag{2.13}$$

i.e. the general terms of the series (2.1) and (2.4) are identical. We accomplish this by induction. Making use of the fact that the differentiation rule of abstract functions with numerical arguments is similar [8] to the rules of ordinary differentiation, we obtain

$$\vartheta^*_{ijkl}\ \partial_{ijkl}(EE^{-1}) = \vartheta^*_{ijkl}\ \partial_{ijkl}\ EE^{-1} + \vartheta^*_{ijkl}\ E\partial_{ijkl}\ E^{-1} = 0 \tag{2.14}$$

Here we have made use of the fact that the operators E and E^{-1} are inverses of each other. By assumption, the operators ϑ^*_{ijkl} and E are commutative, therefore

$$E\vartheta^*_{ijkl}\ \partial_{ijkl}\ E^{-1} = -\vartheta^*_{ijkl}\ \partial_{ijkl}\ EE^{-1} \tag{2.15}$$

This equality proves (2.13) for $n = 1$. We assume now that the identity (2.13) holds for $n = N$ and we prove it for $n = N + 1$. Making use of the assumed commutativity of the operators ϑ^*_{ijkl}, E and E^{-1}, we write the differentiation rule (2.14) in terms of the operator $\vartheta^*_{ijkl}\ \ \partial_{ijkl}$:

$$(\vartheta^*_{ijkl}\ \partial_{ijkl})(EE^{-1}) = E(\vartheta^*_{ijkl}\ \partial_{ijkl})E^{-1} + E^{-1}(\vartheta^*_{ijkl}\ \partial_{ijkl})E \tag{2.16}$$

By straightforward verification we can see that for the powers of the operator $\vartheta^*_{ijkl}\partial_{ijkl}$, acting on a product, the abstract analogue of Leibnitz' formula holds:

$$\begin{aligned}(\vartheta^*_{ijkl}\ \partial_{ijkl})^{N+1}(EE^{-1}) &= E(\vartheta^*_{ijkl}\ \partial_{ijkl})^{N+1}E^{-1} + \\ &+ (N+1)(\vartheta^*_{ijkl}\ \partial_{ijkl})E(\vartheta^*_{mnpq}\ \partial_{mnpq})^N E^{-1} = 0\end{aligned} \tag{2.17}$$

The remaining terms vanish because of the equalities

$$(\vartheta^*_{ijkl}\ \partial_{ijkl})^n E = 0 \qquad (n = 2, 3, \ldots,) \tag{2.18}$$

which follow from the linearity of the operator E, as a function of the parameters E_{ijkl}. We transform the last term of (2.17) in the following way:

$$\begin{aligned}&(N+1)(\vartheta^*_{ijkl}\ \partial_{ijkl}\ E)(E^{-1}E)(\vartheta^*_{mnpq}\ \partial_{mnpq})^N\ E^{-1} = \\ &= (N+1)(\vartheta^*_{ijkl}\ \partial_{ijkl}\ EE^{-1})E(\vartheta^*_{mnpq}\ \partial_{mnpq})^N E^{-1}\end{aligned} \tag{2.19}$$

Since for $n = N$ the identity (2.13) is assumed to be true, we obtain from (2.17) and (2.19)

$$E(\vartheta^*_{ijkl}\ \partial_{ijkl})^{N+1}\ E^{-1} = (-1)^{N+1}(N+1)!\ (\vartheta^*_{ijkl}\ \partial_{ijkl}\ EE^{-1})^{N+1} \tag{2.20}$$

Together with (2.15), the equality (2.20) proves the identity (2.13) and thus theorem 2.1 is proved.

3. The proof of Volterra's principle. Let X_T and Y_T be some one-parameter Banach spaces of vectors with norms $\|\cdots\|_X$ and $\|\cdots\|_Y$, respectively. We will say that $\mathrm{f}(x, t) \in X_T(\Omega \times [0, \infty))$, if for each fixed $t \in [0, T]$ the vector $\mathrm{f}(x, t) \in X_t(\Omega + S)$, and in addition it is continuous with respect to t in the sense of the norm of this space, i.e.

$$\|\mathrm{f}(x, t') - \mathrm{f}(x, t)\|_X \to 0 \quad \text{for } t' \to t \tag{3.1}$$

From condition (3.1) it follows that the norm is a continuous function of the parameter t. In addition, it can be always considered a nondecreasing function of this parameter

$$\|\mathrm{f}(x, t')\|_X \geqslant \|\mathrm{f}(x, t)\|_X \quad \text{for } t' > t,\ x \in \Omega + S \tag{3.2}$$

We assume also that $Y_T \subset X_T$, i.e. the space Y_T is embedded in the space X_T and the inequalities

$$\|\mathrm{f}\|_X \leqslant C\|\mathrm{f}\|_Y, \quad \|\mathrm{f}_{,i}\|_X \leqslant C_i\|\mathrm{f}\|_Y, \quad \|\mathrm{f}_{,ij}\|_X \leqslant C_{ij}\|\mathrm{f}\|_Y \tag{3.3}$$

hold, which are characteristic for the embedding of spaces [9] with additive norms. The constants C, C_i, C_{ij} do not depend on t.

We consider the elastic problem (1.13), (1.14). We assume that this problem is uniquely solvable on the set of vectors $\mathrm{g}(x, t) \in X_T$ and the inverse operator E^{-1} is a bounded operator from the space X_T into the space Y_T

$$\|E^{-1}\mathbf{g}\|_Y \leqslant M\|\mathbf{g}\|_X \tag{3.4}$$

The constant M is independent of the parameter t. The operator E^{-1} maps the space X_T onto a part of the space Y_T. We denote it by Y_T° and we will call it the subspace of the solutions of the elastic problem. Obviously, this is a set of vectors belonging to the space Y_T and satisfying in some sense the equations of the elastic problem and the boundary conditions.

Lemma 3.1. On the set of vectors $\mathbf{u} \in Y_T^\circ(\Omega \times [0, \infty))$ the operators E and $\partial_{ijkl} E$ are bounded operators with values in the space $X_T(\Omega \times [0, \infty))$.

The proof follows from the inequalities (3.3) and the representation (2.6). For example, for the operator E we have the chain of inequalities

$$\|E\mathbf{u}\|_X \leqslant \sum_{i=1}^{3} E_{ijkl}\|\mathbf{u}_{k,lj}\|_X \leqslant \sum_{i=1}^{3} E_{ijkl}C_{klj}\|\mathbf{u}\|_Y = m\|\mathbf{u}\|_Y \tag{3.5}$$

Since $Y_T^\circ \subset Y_T$ and the space Y_T is embedded in the space X_T, we conclude that the operators E and E^{-1} are simultaneously bounded on the subspace Y_T°. In addition, being inverses to each other, they are commutative on this subspace. Then, from the boundedness of the operator E on Y_T° and from the inequalities (3.5) we obtain the existence of the norm $\|E\|_X$. By straightforward verification we can see that in the sense of this norm the operator E is differentiable with respect to the parameter E_{ijkl}, and

$$\partial_{ijkl}E = -\mathbf{e}_i(\cdots)_{k,lj}, \qquad \partial^n_{ijkl} E = 0, \qquad n = 2,3,\ldots \tag{3.6}$$

From Lemma 2.2 we obtain that on the subspace Y_T° the operators E, E^{-1}, $\partial_{ijkl}E$ are commutative.

We proceed now to the investigation of the properties of the viscoelastic problem.

Lemma 3.2. If the tensor $\partial_{ijkl}(t, \tau)$ is continuous and bounded in the norm of the space X_T $(\Omega \times [0, \infty))$, then the operator B_t is bounded in this space.

P r o o f. For the operator A_t we have the following estimate:

$$\| A_t \mathbf{f} \|_X \leqslant \int_0^t \| \vartheta_{ijkl} (t, \tau) \|_X \| \partial_{ijkl} E E^{-1} \mathbf{f} \|_X d\tau \leqslant$$

$$\leqslant M \int_0^t \sum_{i=1}^3 C_{klj} \| \vartheta_{ijkl} (t, \tau) \|_X \| \mathbf{f} \|_X d\tau = M \int_0^t K(t, \tau) d\tau \| \mathbf{f} \|_X (t) \tag{3.7}$$

Here we have made use of the inequalities (3.2), (3.3), (3.4) and we have introduced the bounded function:

$$K(t, \tau) = \sum_{i=1}^3 C_{klj} \| \vartheta_{ijkl} (t, \tau) \|_X \tag{3.8}$$

In addition, for every vector $\mathbf{f} \in X_T$ the vector $A_t \mathbf{f}$ is continuous with respect to t in the norm X_T. Indeed,

$$\| A_{t'} \mathbf{f} - A_t \mathbf{f} \|_X \leqslant \left| \frac{t'}{t} - 1 \right| \| A_t \mathbf{f} \|_X +$$

$$+ \frac{t'}{t} M \int_0^t \sum_{i=1}^3 \left\| \vartheta_{ijkl} \left(t, \frac{t'}{t} \tau\right) - \vartheta_{ijkl} (t, \tau) \right\|_X C_{klj} d\tau \| \mathbf{f} \|_X \tag{3.9}$$

By virtue of the continuity of $\vartheta_{ijkl} (t, \tau)$ and the boundedness of the operator A_t, from (3.9) we obtain

$$\| A_t' \mathbf{f} - A_t \mathbf{f} \|_X \to 0 \quad \text{when } t' \to t, \ \mathbf{f} \in X_T \tag{3.10}$$

which is equivalent to the required continuity. Since A_T acts in the space X_T, its power A_t^n acts in the same space and we have the estimate

$$\| A_t^n \mathbf{f} \|_X \leqslant M^n \int_0^t K_{n-1} (t, \tau) d\tau \| \mathbf{f} \|_X (t) \tag{3.11}$$

where $K_{n-1} (t, \tau)$ is the $(n - 1)$-th iterate of the kernel $K(t, \tau)$. Starting from (3.7), inequality (3.11) can be easily proved by induction.

As a result we obtain for the norm of the operator B_t

$$\| B_t \|_X \leqslant \sum_{n=0}^{\infty} \| A_t^n \|_X \leqslant 1 + \sum_{n=1}^{\infty} M^n \int_0^t K_{n-1} (t, \tau) d\tau \tag{3.12}$$

The series $\sum_{n=1}^{\infty} M^n K_{n-1} (t, \tau)$ is convergent [10] and its sum $R_M (t, \tau)$ is the resolvent of the Volterra operator $I - MK^*$, and thus, B_t represents a bounded operator in the norm of the space X_T. The continuity with respect to the parameter t of the operator B_t follows from the uniform convergence of the series (3.12) with respect to t and from the continuity of the operator A_t

$$\| B_{t'} - B_t \|_X \leqslant \sum_{n=0}^{N} \| A_{t'}^n - A_t^n \|_X + \| r_{t'}^N \|_X + \| r_t^N \|_X < \varepsilon \tag{3.13}$$

since for a sufficiently large index N the remainder of the series (3.12) can be made arbitrarily small independent of t $(\| r_t^N \|_X < \varepsilon / 3)$, and the finite sum tends to zero for

$t' \to t$. Here $\varepsilon > 0$ is an arbitrary small quantity. The lemma is proved.

Theorem 3.1. If the inverse operator E^{-1} of the elastic boundary value problem is an analytic function of the elastic characteristics and does not depend on t, then the formal construction of the inverse operator V_t^{-1} corresponding to the viscoelastic problem with respect to Volterra's principle is always possible and is equivalent to the direct solution.

Proof. We assume, as before, that the operator E^{-1} acts as a bounded operator from the space X_T into the space Y_T. Then, there exists a subspace of solutions of the elastic problem $Y_T^\circ \subset Y_T$, on which the commutativity of the operators E, $\partial_{ijkl}E$, E^{-1} holds (Lemmas 2.2 and 3.1). The operator B_t acts in X_T (Lemma 3.2). Since we have the embeddings $Y_T^\circ \subset Y_T \subset X_T$, one can affirm that the operator B_t acts also in the space Y_T°. Similarly, the operators ϑ_{ijkl}^* under the assumptions of Lemma 3.2, act in X_T and the more so in the subspace Y_T°. Thus, the set of vectors on which the basic operators are commutative is the subspace of solutions of the elastic problem. By the assumption of the theorem, the operator E^{-1}, and consequently the operator E too, do not depend on time, i.e. they are pure coordinate operators. We conclude from here that on the subspace Y_T° the assumptions of Lemma 2.2 and Theorem 2.1 hold. Thus, on the set of solutions of the elastic problem the identity (2.4) holds. The latter means that the formal Taylor expansion represents a bounded operator, whose properties are identical with the properties of the operator B_t (Lemma 3.2). The operators E^{-1} and B_t are commutative in the space X_T. This follows from the commutativity of the operators E^{-1} and A_t, which can be easily established on the basis of representation (2.5), the commutativity of the operators $\partial_{ijkl}E$ and E^{-1} in Y_T° and the independence of E^{-1} from t

$$A_t E^{-1} = \vartheta_{ijkl}^* \ \partial_{ijkl} \ EE^{-1}E^{-1} = \vartheta_{ijkl}^* \ E^{-1}\partial_{ijkl} \ EE^{-1} = E^{-1}A_t \tag{3.14}$$

From all this it follows that the inverse operator V_t^{-1} of the viscoelastic problem in the space X_T admits two equivalent representations

$$V_t^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (\vartheta_{ijkl}^* \partial_{ijkl})^n E^{-1} = \sum_{n=0}^{\infty} (\vartheta_{ijkl}^* \partial_{ijkl} EE^{-1})^n E^{-1} \tag{3.15}$$

The first one corresponds to the application of Volterra's scheme and the second one to the direct solving of the viscoelastic problem. The values of the operator V_t^{-1} belong to the space Y_T. In addition, for the solution of the viscoelastic problem we have the estimate

$$\|\mathbf{u}\|_Y = \|V_t^{-1}\mathbf{f}\|_Y \leqslant M\|B_t\mathbf{f}\|_X \leqslant M\left(1 + \int_0^t R_M(t, \tau)\,d\tau\right)\|\mathbf{f}\|_X. \tag{3.16}$$

Theorem 3.2. If the inverse operator E^{-1} of the elastic problem depends on time, then the inverse operator V_t^{-1} of the viscoelastic problem has a unique representation which follows from its direct solving. The formal application of Volterra's principle reduces to a nonidentical operator.

The proof follows from the fact that in the case when the operator E^{-1} depends on t, the necessary conditions for the equivalence of the two schemes for solving the visco-elastic problem (Sect. 2) are violated. In addition, the conditions of Theorem (2.1), which are necessary for the existence of the identity (2.4), are violated. Indeed, assume that the identity (2.4) holds. Then, the identities (2.13) hold, and consequently (2.15) holds. Multiplying the latter by the operator E^{-1} on the left and then by E on the right, we obtain

$$\vartheta^*_{ijkl} E \partial_{ijkl} E^{-1} + E^{-1} \vartheta^*_{ijkl} \partial_{ijkl} E = 0 \tag{3.17}$$

Subtracting this from the same identity (2.15), we obtain

$$(E\vartheta^*_{ijkl} - \vartheta^*_{ijkl} E) \partial_{ijkl} E^{-1} + (\vartheta^*_{ijkl} E^{-1} - E^{-1}\vartheta^*_{ijkl}) \partial_{ijkl} E = 0 \tag{3.18}$$

Since

$$\partial_{ijkl} E^{-1} = - E^{-1}E^{-1} \partial_{ijkl} E \tag{3.19}$$

and the operator $\partial_{ijkl}\ E$ is not identically zero, from (3.18) we obtain

$$(E\vartheta^*_{ijkl} - \vartheta^*_{ijkl}E) E^{-1} - (\vartheta^*_{ijkl} E^{-1} - E^{-1} \vartheta^*_{ijkl})E = 0 \tag{3.20}$$

Hence it follows that

$$E\vartheta^*_{ijkl} - \vartheta^*_{ijkl} E = 0, \qquad \vartheta^*_{ijkl} E^{-1} - E^{-1}\vartheta^*_{ijkl} = 0 \tag{3.21}$$

i.e. the operators E and E^{-1} commute with the operators ϑ^*_{ijkl}. The obtained contradiction proves the theorem. The proof for the representation of the inverse operator of the viscoelastic problem is obtained only by its direct solving and it is given by the series (1.17).

4. Applications. A further concretization of the formulation of the boundary value problem of viscoelasticity leads to the necessity of considering it in specific function spaces. We consider two widespread approaches.

The investigation of smooth solutions (the classical approach) is carried out appropriately in the Hölder spaces $C^{m+\alpha}(\Omega)$ (see [9]). For the boundary value problem of viscoelasticity in terms of displacements such an approach is given in [11]. In the case of a regular domain Ω, under the condition of positive definiteness of the tensor E_{ijkl} and continuity of the tensor $\vartheta_{ijkl}(t, \tau)$, for each vector $\mathbf{f}(x, t) \in C^{\alpha}_{T}(\Omega \times [0, \infty))$ there exists a unique vector $\mathbf{u}(x, t) \in C^{2+\alpha}_{T}(\Omega \times [0, \infty))$ satisfying the equations and the boundary conditions (1.12) of the viscoelastic problem. Selecting for the spaces X_T and Y_T the spaces C^{α}_{T} and $C^{2+\alpha}_{T}$, respectively, ($C^{m+\alpha}_{T}$ is a Banach space), with the usual [9] norms $\|\cdot\|_{\alpha}$ and $\|\cdot\|_{2+\alpha}$, we can verify that all the assumptions hold.

As a result we obtain

Corollary 1. Under the assumptions of Theorem 3.1, the smooth solutions of the viscoelastic problem can be always obtained by the formal application of Volterra's principle.

Corollary 2. If the inverse operator of the elastic boundary value problem depends on time, then for the construction of the smooth solutions of the corresponding viscoelastic problem, Volterra's principle is not applicable. They can be obtained by direct solution using the second scheme (Sect. 2).

An attempt to weaken the conditions on the right-hand sides of the differential equations and the boundary values of the viscoelasticity problem, leads to the necessity of introducing generalized solutions [9]. In this connection the fundamental equations are satisfied in the weak (integral) sense, while the derivatives with respect to the coordinates are understood to be generalized derivatives. The existence and uniqueness of the generalized (weak) solution of the boundary value problem of viscoelasticity in terms of displacements have been proved in [12], moreover, if $\mathbf{f} \in L_2$, then $\mathbf{u} \in W_2^1$. Here L_2 is the space of square summable functions and W_2^1 is Sobolev's space

[9]. Making use of the definition of the norms in these spaces and also of the embedding theorem [9], we can see that the spaces X_T and Y_T can be identified with the spaces L_r and W_2^1, respectively.

Hence we obtain:

Corollary 3. If the inverse operator of the elastic problem, regarded in the weak sense, satisfies the conditions of Theorem 3.1, then the generalized solution of the corresponding viscoelastic problem can be obtained by Volterra's principle.

Corollary 4. Under the conditions of Theorem 3.2, the construction of the generalized solutions of the viscoelastic problem is possible only by using the second scheme regarded in the weak sense. Thus, the fundamental criterion for the practical use of Volterra's principle is the dependence or the independence of time of the inverse operator of the corresponding elastic boundary value problem. Even in the case of the separation of the operations with respect to coordinates and time in the equations, the inverse operator may turn out to be a function of time, at the expense of the fact that the boundary conditions are formulated on moving surfaces. As examples we have numerous problems of viscoelasticity with moving supports, annihilating (ablating) surfaces and so on. The procedure of solving such problems using the second scheme allows the use of electronic computers with efficiciency.

In conclusion we note that the obtained results are qualitatively valid in the case of the second and the third fundamental boundary value problem of viscoelasticity.

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EQUATIONS OF MOTION OF NEMATIC LIQUID-CRYSTAL MEDIA

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E. L. AERO and A. N. BULYGIN
(Leningrad)
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Equations of motion for nematic liquid-crystal media in a magnetic field and also the equations of thermal conductivity are obtained. Together with the condition of continuity and the equation of state, these relationships determine the fields of nine quantities which characterize the nematic fluid: densities, pressures, temperatures, basis vectors of the local axis of anisotropy, rates of collective rotations of molecules near their "long" axes, and vectors of translational velocity. Initial conditions and boundary conditions are formulated. Special cases are examined: equilibrium of the medium in a homogeneous magnetic and temperature field, disinclinations, orientational boundary layer, and also the flow in a flat capillary in a magnetic field and the drag of fluid by a rotating magnetic field. Based on obtained results, an explanation is given for a number of effects which have been discovered experimentally earlier.

Liquid crystals occupy on the thermodynamic scale of states an intermediate (mesophase) position between anisotropic crystals and isotropic liquids. Two fundamental varieties of mesophases exist: the smectic and the nematic. In the liquid crystal medium of the smectic type the one-dimensional long-range coordination structure is preserved. The molecules are organized in regularly spaced parallel monolayers. In the medium of the nematic type the long-range order is completely absent in the spatial arrangement of molecules, just as in the ordinary liquid. However, in contrast to a liquid and in similarity to a solid crystal the long-range order of orientation for the "long" molecular axes is preserved. The orientational order is characterized in each point of the medium by the axis of mean molecular orientation. This axis is simultaneously the local axis of symmetry of the medium.

In their mechanical properties the nematic media are quite close to liquids. Experiments show [1, 2] that the behavior of nematic liquids in a force field, a temperature field, a magnetic field, and an electrical field has a number of anomalies (anisotropy of viscosity, scale effect, orientation in hydrodynamic flow, drag of the medium by a rotating magnetic field, and others.)

The peculiar combination of mechanical properties makes liquid crystal media interesting objects for investigation from the point of view of continuum mechanics. At the present time the hydrostatic theory [3 - 9] is the most developed. In papers [10 - 12] linear hydrostatics is examined with consideration of thermal conductivity and effects of rotational viscosity. The hydrodynamic theory which takes into account elastic and thermal effects in a magnetic field is just being